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THE KALUZA-KLEIN THEORY

by

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ABSTRACT

In an early attempt to unify two fundamental forces of nature electromagnetism and gravity, Theodore Kaluza of Russia and Oskar Klein of Sweden (1921 and 1926 respectively) succeeded in showing a remarkable connection between these rather different interactions. This was achieved by adding to the metric tensor a fifth coordinate, being surpressed however, by demanding that the metric be independent of this new coordinate. The g_{oi} is then interpreted as the electromagnetic vector-potential and g_{oo} is assumed to be constant. In trying to show how this unification can be done, there arises the problem that the Einstein tensor

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R$$

depends on the $g_{oi} = A_i$, which is not a measurable quantity. However, in forming the scalar action, one does indeed find the proper fielddensity $R - \frac{1}{4}F^{lm}F_{lm}$ and so the situation is saved in forming the fieldequations by the variational principle.

The rôle of the cornercomponent is investigated in an analogous way by putting the $A_i = 0$ and $g_{oo} = f(x)$, and fieldequations are obtained from this in a rescaling. Finally, a generalisation of the vacuum metric is discussed (now a n x n tensor), and it is argued (but not proved !) that this is compatible with the general variational principle

$$\delta S = \delta \int P^{(n)} \sqrt{\gamma}\, d\Omega = 0 \quad , \quad P^{(n)} = \gamma^{ik} P_{ik}$$

where $P = f(\gamma_{ik}, \frac{\partial \gamma_{ik}}{\partial x^l})$ and $i,k = 1 \ldots n$

INTRODUCTION

One of the hottest problems of modern physics is coming to terms with the four fundamental forces of nature: The strong force, acting between nuclei in atoms and still not understood properly, the weak interaction, responsible for the decay of the neutron , the electromagnetic forces, and, finally, the feeblest of them all, gravity. The weak and electromagnetic are now believed to be united in the Weinberg-Salaam theory, and has been confirmed by Rubbia and his team at CERN. Current work on incorporating the strong force in this theory seems promising, and would be testable in that it predicts the decay of the proton. The fourth interaction, about 4x10 the strength of the strong force, has sofar elused all attempts of unification. The main reason is that gravity in its basic structure is fundamentally different from the quantum mechanical models that are believed to describe the other forces. No one has yet successfully quantisized the gravitational field, and it seems to be a very difficult task.

The concept of unification, though, is not a new one. As early as 1914, Gunnar NordstrÖm at Helsinki university outlined a fivedimensional theory attempting to unify these forces. It had to be abandoned though, since it failed to account for the bending of light, as predicted by Einsteins Gravitational theory.

What here follows is an attempt to reproduce and explain the theory of Theodore Kaluza and Oskar Klein, which , quite successfully, ties these rather different interactions together. A brief account of the two different theories is given in order to substantiate the Kaluza-Klein theory.

PREFACE TO GRAVITY

Ever since the day when Euclid developped his axiomatic theory of geometry, mathematicians have wondered at the difference in the complexity of the postulates he made to support the geometry. There are five postulates, all which are formulated in one line, with one exception. This exception, known as the " parallell postulate ", states that (1) " Whenever a line

(1) A. Pogorelev "lectures on the foundations of Geometry" 1966 P. Nordhoff Ltd.

intersects two other lines, forming with the latter samesided interior angles, the sum of which is less than two right angles (90°), these lines intersect on that side from which this sum is less than two right angles. ".

This sounds a bit overworked, and this was realised by greek mathematicians contemporaneous with Euclid. Over the centuries there were various attempts to prove this postulate by reformulating the other four in a way from which the fifth postulate could be derived. Saccheri (1733) and Lambert (1766) pursued attempts in this direction, all of which failed. It was however realised that an equvalent version of this postulate was " Through a point not on a line there passes not more than one line that is parallell to the given line. ". The first real breakthrough in this matter originated from the great russian mathematician Lobachevsky (1793 - 1856), nowadays considered as the founder of non-Euclidian geometries. His version of this - notorious - postulate was " Through a point not on a line in the plane, there pass two lines which do not intersect the given line." . To his and his contemporaries surprise he found no contradiction with the other postulates, and so he had formulated a different geometry.

Later in the 17:th century, Gauss and Bolyai arrived at similar but less elaborate results. Gauss, still later, derived a formula which describes the curvature of a surface in terms of a reciprcal of a product of two numbers, simply called the curvature radii. He also managed to prove that a both sufficient and a necessary condition for a surface to be flat - or Euclidian - was the nullification of this number called the scalar curvature. The work on these peculiar geometries culminated in a thesis by G. Riemann (1854) who gave a general theory of curved surfaces valid in any number of dimensions - these general surfaces are called manifolds or simply hypersurfaces - whose application in physics was to become reality 60 years later in Eistein's beautiful theory. With Riemann, the branch of geometry called differential geometry was born.

PREFACE TO ELECTROMAGNETISM

In the 1840:ies, Faraday launced an attack at the problem of explaining the mysteroius electric and magnetic forces that had been observed for a long time, but that sofar had defied a natural explanation. He asserted that these could be seen as " fieldlines ", a kind of bended grid that extended all over space and that in itself possessed energy and which acted on charged particles as a force; that is, a forcefield. These ideas were set in mathematical form by Maxwell, who, as a curious fact derived them using purely mechanical arguments, and as a nasty sideeffect, introduced the notorious " ether ". The equations, as seen in most modern treatises on electricity were cast in vectorform by Heaviside, thereby revealing the wavenature of these fields. By the beginning of 1910, both of the cornerstones of modern physics were laid in quantum theory and the theory of special relativity. It was found, that since time really was no separate property in itself all physical entities had to be given in four coordinates, space and time; hence the word " spacetime " was invented. A remarkable property of the Lorentztransformations of special relativity is that Maxwells equations are invariant under these. Newtons theory, however had to be revised in order to fit into the new theory. This is basically because a force in Newtonian theory acts instantaneously, whereas in Maxwells theory, the force fields travel at the speed of light (which is fast enough, but not infinitely fast.). In Minkowski's version of relativity, the Lorentzboost - boost being a physicist's sloppy name for a transformation - could be seen as a rotation in a fourdimensioal space with coordinates

$(-ct, x, y, z)$

Also the invariance of the speed of light implies the invariance of the quantity

$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$

called the interval (= 0 along the path of a light ray) and where dT = ds/c is called the proper time. By this time another striking quality was found: In applying the principle of least action to the following " Lagrangian " put in a vectorform

$$S = -\int (mc\,ds + A_i\,dx^i) = -\int \left(mc\sqrt{dx_i dx^i} + A_i dx^i\right) \qquad (1)$$

one derives easily the equations of motion in the following notation:

$$mc\frac{du^i}{ds} = q F^i{}_k u^k \qquad (2)$$

The quantity $F_{ik} = \partial A_i/\partial x^k - \partial A_k/\partial x^i$ is called the elctromagnetic fieldtensor and the A_i is denoted the vectorpotential. It can be shown that Maxwells equations are reduced to the following relations:

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{E}}{\partial t} \qquad \Rightarrow \qquad (3a)$$

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad (3b)$$

$$\frac{\partial F_{il}}{\partial x^k} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{lk}}{\partial x^i} = 0 \qquad (3c)$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \qquad (4a)$$

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \qquad \Rightarrow \qquad (4b)$$

$$\frac{\partial F^{ik}}{\partial x^k} = \mu_0 J^i \qquad (4c)$$

TThe vectorpotential itself is not a measurable quantity; only derivatives of this otherwise artificial object are relevant. The vector is related to the fields as follows:

$$\underline{B} = \underline{\nabla} \times \underline{A}$$

$$E = -\nabla\varnothing - \frac{\partial \underline{A}}{\partial t} \qquad (5)$$

and the fieldtensor as

$$F_{ik} = \begin{pmatrix} 0 & -E_1/c^2 & -E_2/c^2 & -E_3/c^2 \\ E_1/c^2 & 0 & B_3 & -B_2 \\ E_2/c^2 & -B_3 & 0 & B_1 \\ E_3/c^2 & B_2 & -B_1 & 0 \end{pmatrix} \qquad (6)$$

TOWARDS UNIFICATION

In 1915, ten years after his special relativistic theory, he emerged with the general version. The special version was confined to describing events in uniformly moving - inertial - reference systems, whereas the general theory is applicaLle to all systems of reference. The mathematical tools necessary bring the theory to life had been around ever since the days of Riemann and the Levi-Civita tensor calculus provided a powerful tool in treating the theory in a convenient way. With the help of mathematicians like Grassmann he was able to formulate gravity using the simple defining properties of tensors:

$$R_{ik} = \Lambda^{\bar{i}}_{i} \Lambda^{\bar{m}}_{k} R_{\bar{i}\bar{m}} \quad , \Lambda^{\bar{i}}_{i} = \frac{\partial x'^{l}}{\partial x^{i}} \qquad (7)$$

This is called the principle of covarianace, and expresses precisely the liberty of applying the equations in a general reference system - or frame - . He generalised Minkowskis spacetime in introducing a generalised line element as

$$ds^2 = g_{ik} dx^i dx^k \qquad k,i = 0\ldots3 \qquad (8)$$

Where the g_{ik} is called the metric tensor. This tensor has the the rôle of gravitational potentials; instead of having one fieldequation for the newtonian potential - Poissons equation - or

$$\nabla \phi = 4\pi\rho \qquad (9)$$

one has now a number of independent relations (14 of them in the most general case) intimately related to Riemanns theory of manifolds according to which the properties of the latter is described by a rank 4 tensor:

$$R^{i}_{klm} = \frac{\partial \Gamma^{i}_{km}}{\partial x^{l}} - \frac{\partial \Gamma^{i}_{kl}}{\partial x^{m}} + \Gamma^{i}_{nl}\Gamma^{n}_{km} - \Gamma^{i}_{nm}\Gamma^{n}_{kl} \qquad k,i,l,m = 0\ldots3 \qquad (10)$$

TThe symbol Γ^{i}_{kl} is called Christoffel symbols or connection coefficients, and they arise in the Riemannian theory in taking the derivative of a vector or a tensor on the manifold. Since the value of a vector generally depends on its location on the manifold - not the case in euclidian space - one must, in taking the difference between two vectors infinitesimally separated, subject the vector at one point to an operation called parallell transport. The Γ^{i}_{kl} can be seen as linear operator whose task is to perform this operation. In tensor language

$$\delta A^i = \Gamma^{i}_{mk} A^m dx^k \qquad (11)$$

which must be added to the ordinary derivative to obtain an expression valid in all frames. The quantity

$$\frac{DA^i}{\partial x^m} = \frac{\partial A^i}{\partial x^l} + \Gamma^{i}_{ml} A^m \qquad (11a)$$

is called a covariant derivative, and similar formulas can be derived for covariant and contravariant tensors. The connection coefficients are not tensors, since they vanish in a locally flat frame (it can be shown that there is always a locally flat frame in which $dg_{ik}/dx^l = 0$), and according to (7) they should vanish in all frames which contradicts (11).

The usefulness of the connection coefficients lies in their relation to the metric tensor. In tensor calculus it shown that one can " raise & lower " tensors using the g_{ik} as $A^i = g^{im} A_m$, $A_l = g_{lm} A^m$. As a result of this one shows easily that

$$\frac{Dg_{ik}}{\partial x^l} = 0$$

From a general formula for covariant differentiation it turns out that

$$\Gamma^l_{kl} = \frac{g^{lm}}{2}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{lm}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right) \qquad (12)$$

Einstein argued that a most natural description of gravity must be linear combination of second rank tensors R_{ik} describing the properties of spacetime. The source of the curvature of spacetime is given by another second rank tensor called the energy-momentum tensor, having the same function as the r.h.s. of eq. (9). This tensor can be defined according to the principle of least action, or

$$\delta S = \delta \int L\left(g^{ik}, \frac{\partial g^{ik}}{\partial x^l}\right) d\Omega = 0 \qquad (13)$$

Applying the Euler-Lagrange equations to this yields*

$$T_{ik}\sqrt{g} = \frac{\partial(\sqrt{g}L)}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\left(\frac{\partial(\sqrt{g}L)}{\partial\left(\frac{\partial g^{ik}}{\partial x^l}\right)}\right) \qquad (14)$$

This tensor is analogous to the stress tensor in elasticity-theory, and the fourmomentum - containg information on both energy and momentum - is

$$P^i = \int T^{ik} dS_k \quad \text{(taken over a hyper surface)} \qquad (15)$$

* Note that eq (14) does <u>not</u> imply $T_{;k} = 0$, but $DT_{ik}/x^l = 0$. See Landau/Lifshitz "Classical Theory of Fields, footnote p 272.

The tensor has the following meaning:

$$T^{ik} = \left(\begin{array}{c|c} E & S_\alpha/c \\ \hline S_\alpha/c & Q_{\alpha\beta} \end{array}\right) \tag{16}$$

E = energy density, S_α = amount of energy transferred through a unit surface per unit time and $Q_{\alpha\beta}$ = amount of momentum transferred through a unit surface perunit time.

In purely intuitive manner, Einstein reasoned that the eq's governing gravity ought to look like

$$R_{ik} - \frac{1}{2} g_{ik} R = k T_{ik} \tag{17}$$

where the $R_{ik} = R^l_{ilk}$ and this contracted tensor is called the Ricci tensor. Weakfield considerations give the value of the constant k and it turns to be

$$k = \frac{8\pi G}{c^4}$$

For electromagnetic fields the energymomentum tensor is

$$T_{ik} = \frac{1}{4} g_{ik} F^{lm} F_{lm} - F_i{}^l F_{lk} \tag{18}$$

Which is produced by applying (14) to

$$S = -\frac{1}{4} F^{lm} F_{lm}$$

Around the 1920:ies people began too see connections between Einsteins general theory of relativity and Maxwells theory and different approaches were tried by H. weyl, H. Thirring and, as mentioned before, G. NordstrÖm.
It was, however an unknown privatdocent in Russia, Theodore Kaluza who managed to bring some light on the problem.

THE KALUZA-KLEIN THEORY

In 1919, as disclosed in series of letters to Einstein, he came up with the idea of adding an extra dimension to the metric. This fifth coordinate is however surpressed in demanding that the 5-d metric be independent of the extra dimension. Furthermore in putting γ_{00} = const, one gets the following metric:

$$\gamma_{ik} = \left(\begin{array}{c|c} a & \gamma_{i0} \\ \hline \gamma_{0k} & \gamma_{ik} \end{array}\right) \tag{20}$$

The reason for this ansatz is the following:
In calculating the connection with this metric, one of the symbols are:

$$\Gamma_{0ik} = \tfrac{1}{2}(\partial g_{0i} / \partial x^k - \partial g_{0k} / \partial x^i) = \tfrac{1}{2}(g_{0i,k} - g_{0k,i}) \qquad (21)$$

where $\partial g_{0k} / \partial x^i = g_{0k,i}$ and similarly, as a time and space saving measure, $DR_{0k} / \partial x^i = R_{0k;i}$.

Now, (21) looks remarkably like the definition of the electromagnetic field tensor!

What Kaluza then did was to show that this assuumption brings out the correct equations of motion in the weakfield limit, with some necessary assumptions of rôle of the fifth coü ordinate. The work was published in 1921 by Einstein, having decided after twoyear correspondance with Kaluza , that the treatise was consistent.

In 1926, Oskar Klein of Sweden published a simliar work in which he tries to take quantum mechanical effects into account He also derived the results of Kaluza in a different manner, in starting directly from the actionprinciple due to the famous mathematician David Hilbert. The latter presented a rigorous derivation of Einstein's fieldequations as

$$\delta S = \delta \int \sqrt{g}\,(R + L)\, d\Omega = 0 \qquad (22)$$

R is the scalar curavture, $R = g^{ik} R_{ik}$, and L is the action of the source of the field. In applying formula (14) to this, it can be shown that $\delta R_{ik} = 0$ (see e.g. Landau/Lifshitz p. 275) and leads to eq. (17).
What is remarkable about this theory is that both the 4-d gravitational scalar curvature + the electromagnetic scalar drops out of the calculation with L=0.
This can be shown as follows:

Assuming that the 5-d metric is independent of x leads to the following possible coordinate transformation:

$$x^0 = x^{\bar{0}} + f_0(x^{\bar{i}}) \qquad \bar{i} = 1-4$$
$$x^i = f_i(x^{\bar{i}}) \qquad i = 1-4 \qquad (23)$$

As a consequence

$$\gamma_{00} = \Lambda^{\bar{\mu}}_{0} \Lambda^{\bar{\nu}}_{0} \gamma_{\bar{\mu}\bar{\nu}} = \delta^{\bar{\mu}}_{0} \delta^{\bar{\nu}}_{0} \gamma_{\bar{\mu}\bar{\nu}} = \gamma_{00} = \text{const.}$$

It's also easy to show that the following quantity is invariant under (23)

$$ds^2 = \left(\gamma_{ik} - \frac{\gamma_{0i}\gamma_{0k}}{\gamma_{00}}\right) dx^i dx^k \tag{24}$$

(See appendix III).

One can now put the metric in the following convenient form:

$$\gamma_{ik} = \left(\begin{array}{c|c} a & abA_k \\ \hline abA_i & g_{ik} + ab^2 A_i A_k \end{array}\right) \quad \gamma^{ik} = \left(\begin{array}{c|c} \frac{1}{a} + b^2 A^m A_m & -bA^k \\ \hline -bA^i & g^{ik} \end{array}\right) \tag{25}$$

b = constant, a = γ_{00} .

The connection coefficients are now, according to (12):*

$$\Gamma^{\mu}_{00} = \Gamma^{0}_{00} = 0 \qquad \Gamma^{0}_{0\alpha} = -\frac{ab^2}{2} A^{\mu} F_{\mu\alpha} \qquad \Gamma^{\alpha}_{0\beta} = \frac{ab}{2} F^{\alpha}{}_{\beta}$$

$$\Gamma^{\alpha}_{\beta\gamma} = \hat{\Gamma}^{\alpha}_{\beta\gamma} + \frac{ab^2}{2}\left(A_{\beta} F^{\alpha}{}_{\gamma} + A_{\gamma} F^{\alpha}{}_{\beta}\right) \tag{26}$$

$$\Gamma^{0}_{\beta\gamma} = -bA_{\mu}\hat{\Gamma}^{\mu}_{\beta\gamma} - \frac{ab^3}{2} A^{\mu}\left(A_{\beta} F_{\mu\gamma} + A_{\gamma} F_{\mu\beta}\right) + \frac{b}{2}\left(A_{\beta,\gamma} + A_{\gamma,\beta}\right)$$

The reason for the departure from the the simple connection of Kaluza is that these symbols are given in a " contravariant " form, and since $\Gamma_{abc} = g_{ad}\Gamma^{d}_{bc}$, these are by necesssity more complicated. Seing these symbols, one is ready to tear ones hair in despair. Being of stubborn nature, however, one sets one sets to the tedious task of feeding these into the Riemann tensor. Working in an orthonormal frame, this gives:

$$R_{km} = \hat{R}_{km} - \frac{1}{2} ab^2 F^{a}{}_{k} F_{am} + \frac{ab^3}{2}\left(A_k F^{l}{}_{m;l} + A_m F^{l}{}_{k;l}\right) + \frac{a^2b^4}{4} A_k A_m F^{al} F_{al}$$

$$R_{0k} = \frac{ab}{2} F^{l}{}_{k;l} + \frac{a^2b^3}{4} A_k F^{al} F_{al} \tag{27}$$

$$R_{00} = \frac{a^2b^2}{4} F^{al} F_{al}$$

* there is another way of doing this: In applying Euler-Lagrange equations to $L = \frac{1}{2} g_{ik} \dot{x}^i \dot{x}^k$ this producues the eq's of motion $g_{il}\ddot{x}^l + \Gamma_{ilm}\dot{x}^l\dot{x}^m = 0$. Thus the Γ_{ilm} are in "covariant" form.

Now, it's interesting to note that in forming the Einstein tensor

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R$$

one does not obtain the energy-momentum tensor as in (18). We nevertheless continue and form the curvature scalar as

$$R = g^{km} R_{km} + 2\gamma^{0k} R_{0k} + \gamma^{00} R_{00} \tag{28}$$

and this does indeed turn out to be

$$\hat{R} - \frac{1}{4} ab^2 F^{\ell m} F_{\ell m}$$

which is consistent with (18).

As a curious note, one might add that in putting the A -0, then in forming the Einstein tensor, this is consistent with (18), and furthermore, in forming G this is simply

$$\frac{ab}{2} F^{\ell}{}_{k;\ell} = 0 \Rightarrow \frac{\partial(\sqrt{g}\, F^{k\ell})}{\partial x^{\ell}} = 0$$

which is eq. (4c) on a curved manifold.

The reasoning above is, of course, highly dubious. First one assumes a 5-d metric that has off diagonal components, interpreted as vector potentials. One then hopes that these will appear in the Einstein tensor only acted upon by a differential operator (the covariant or the ordinary derivative). It is found, to ones dismay, that this is not the case. This is peculiar in at least to ways: First of all, the vectorpotential it self is not a measurable quantity. Secondly, the G_{ik} should be gauge invariant, and so all auxiliary fields should be absent.

On the other hand, the curvature scalar drops out nicely from the calculations, indicating that

a) only the contracted Ricci tensor is gauge invariant, or

b) there is a slip somewhere in the calcuations.

If b) is correct, then the error probably lies somewhere in the connection. This belief is better explained in showing the christoffel symbols in the " covariant " form. These are:

$$\Gamma_{000} = \Gamma_{00k} = \Gamma_{k00} = 0 \qquad \left(\text{since } \frac{\partial}{\partial x^0} = 0\right)$$

$$\Gamma_{0\ell m} = \frac{1}{2} ab (A_{m,\ell} + A_{\ell,m}) \qquad \Gamma_{m0k} = \frac{ab}{2} F_{mk}$$

(The words "covariant" and "contravariant" are actually without meaning since the Γ^{i}_{kl} do not form a tensor. There should be no ambiguity though, since it's a matter of convention.)

Since the Riemann tensor generally in the literature is given in terms of "contravariant" symbols, these have to be turned into the correct form. This is done by

$$\Gamma^{m}_{kl} = \gamma^{mi}\Gamma_{ikl} + \gamma^{m0}\Gamma_{0kl}$$

If this is valid, then there really shouldn't be any difficulties in the rest of the calculations. If this is not true, it may be possible that greater caution has to be taken in going from Γ_{klm} to Γ^{k}_{lm}. Aproper treatment would then be ,as i see it to calculate the cuvature tensor in a pure covariant form. This calculation has, nevertheless, produced the correct scalar actions, from which it is a simple matter to derive the field equation from the variational principle.

THE "CORNER" COMPONENT

Another feature of interest is the rôle of the "corner" component, which can be dealt with conveniently by putting the $A_i = 0$ and $\gamma_{00} = f(x)$, i.e.

$$\gamma_{ik} = \left(\begin{array}{c|c} \varphi(x) & 0 \\ \hline 0 & g_{ik} \end{array}\right) \quad \gamma^{ik} = \left(\begin{array}{c|c} \varphi(x)^{-1} & 0 \\ \hline 0 & g^{ik} \end{array}\right) \tag{34}$$

The connection (a lot more simple this time) is:

$$\Gamma^{0}_{00} = \Gamma^{0}_{\mu\nu} = \Gamma^{\mu}_{\nu 0} = 0 \qquad \Gamma^{\mu}_{00} = -\frac{1}{2}\varphi^{,\mu}$$
$$\Gamma^{0}_{0\mu} = \frac{1}{2\varphi}\varphi_{,\mu} \qquad \Gamma^{\mu}_{\nu\gamma} = \hat{\Gamma}^{\mu}_{\nu\gamma} \tag{35}$$

and the Riccitensors are

$$R_{km} = -\frac{1}{2\varphi}\varphi_{,km} + \frac{1}{4\varphi^2}\varphi_{,k}\varphi_{,m} + R_{km}$$
$$R_{00} = -\frac{1}{2}\varphi^{,l}{}_{;l} + \frac{1}{4\varphi}\varphi^{,l}\varphi_{,l} \tag{36}$$

and

$$R = \hat{R} - \frac{\varphi^{,a}{}_{;a}}{\varphi} + \frac{1}{2\varphi^2}\varphi^{,a}\varphi_{,a} \tag{37}$$

Partial integration of second term and using the formula for the four divergence

$$\varphi^{,a}{}_{;a} = \frac{1}{\sqrt{g}}(\sqrt{g}\,\varphi^{,a})_{,a} \tag{38}$$

reveals that the terms actually cancel and so

$$S = \int \sqrt{g}\,\sqrt{\varphi}\, d\Omega \tag{39}$$

The factor $\sqrt{\varphi}$ is a bit of a bore, but can be gotten rid of re-scaling the metric as

$$\gamma_{ik} = \left(\begin{array}{c|c} \varphi^{p+1}(x) & 0 \\ \hline 0 & \varphi^{p} g_{ik} \end{array}\right) \qquad \gamma^{ik} = \left(\begin{array}{c|c} \varphi^{-1-p}(x) & 0 \\ \hline 0 & \varphi^{-p} g^{ik} \end{array}\right) \tag{40}$$

and the connection :

$$\Gamma^{0}_{00} = \Gamma^{0}_{\mu\nu} = \Gamma^{\mu}_{\nu 0} = 0 \qquad \Gamma^{\mu}_{00} = -\frac{(p+1)}{2}\varphi^{,\mu}$$

$$\Gamma^{0}_{0\mu} = \frac{(p+1)}{2\varphi}\varphi_{,\mu} \qquad \Gamma^{\mu}_{\nu\gamma} = \hat{\Gamma}^{\mu}_{\nu\gamma} + \frac{p}{2\varphi}\left(\delta^{\mu}_{\nu}\varphi_{,\gamma} + \delta^{\mu}_{\gamma}\varphi_{,\nu} - g_{\gamma\nu}\varphi^{,\mu}\right) \tag{41}$$

S witching the mind into tensor mode again reveals that

$$R_{km} = -\frac{1}{6}\frac{\varphi_{,k}\varphi_{,m}}{\varphi^2} - \frac{1}{6}\frac{g_{km}\varphi^{,s}\varphi_{,s}}{\varphi^2} + \frac{1}{6}\frac{g_{km}\varphi^{,a}{}_{;a}}{\varphi} \tag{42}$$

$$R_{00} = -\frac{1}{3}\varphi^{,a}{}_{;a} + \frac{1}{3}\frac{\varphi^{,a}\varphi_{,a}}{\varphi} \tag{43}$$

The last term in (42) can be integrated by using (38) and

$$S = \frac{1}{6}\int \sqrt{g}\, g_{km}\frac{\varphi^{,a}{}_{;a}}{\varphi}\, d\Omega = \frac{1}{6}\int g_{km}\frac{(\sqrt{g}\,\varphi^{,a})_{,a}}{\varphi}\, d\Omega$$

Working in a Minkowski frame in which $g_{km,\lambda} = 0$ yields

$$S = \frac{1}{6}\int \sqrt{g}\, g_{km}\frac{\varphi^{,a}\varphi_{,a}}{\varphi^2}\, d\Omega \Rightarrow R_{km} = -\frac{1}{6}\frac{\varphi_{,k}\varphi_{,m}}{\varphi^2} \tag{44}$$

Finally the action is $R = \hat{R} - \frac{1}{2}\varphi^{,a}\varphi_{,a} + \frac{1}{3}\varphi^{,a}{}_{;a}$. Integrating the second term

$$R = \hat{R} - \frac{1}{6}\frac{\varphi^{,k}\varphi_{,k}}{\varphi^2} \tag{45}$$

The energy-momentum tensor, as formed by the Ricci tensor above, is then

$$G_{ik} = \frac{1}{6\varphi^2}\left(\frac{1}{2}g_{ik}\varphi^{,l}\varphi_{,l} - \varphi_{,i}\varphi_{,k}\right) \tag{46}$$

which is readily derived by varying $-\frac{1}{6}\delta\left(\frac{\varphi^{,k}\varphi_{,k}}{\varphi^2}\right)$ with respect to the g_{ik} and $g_{ik,m}$.

It can be shown that in performing a calculation in which the entire metric depends on ordinary space-time, one gets a super-position of the Kaluza-Maxwell field and the "corner"-field, the action then being of the type *

$$S = \int \sqrt{g}\left(\hat{R} + A F^{lm} F_{lm} + B \frac{\varphi^{,l}\varphi_{,l}}{\varphi^2}\right) d\Omega \tag{47}$$

In "rescaling" the Kaluza-Klein metric in putting a = 1, results in $A = -\frac{b^2}{4}$, $B = -\frac{1}{6}$. By virtue of the variational principle again the following fieldequations emerge:

$$G_{ik} - \frac{b^2}{2}\left(F^l{}_k F_{lm} - \frac{1}{4} g_{km} F^{la} F_{la}\right) - \frac{1}{6\varphi^2}\left(\varphi_{,k}\varphi_{,m} - \frac{1}{2} g_{km}\varphi^{,a}\varphi_{,a}\right) = 0 \tag{48}$$

In the case of an empty spacetime where $F_{ik} = 0$

$$G_{ik} = \frac{1}{6\varphi^2}\left(\varphi_{,k}\varphi_{,m} - \frac{1}{2} g_{km}\varphi^{,a}\varphi_{,a}\right) \tag{49}$$

In the general principle of variations, one can assume a metric in as many dimensions as one likes; in varying the action in the usual way one obtains

$$\delta S = \delta\int \gamma^{ik} P_{ik}\sqrt{\gamma}\, d\Omega = \int \left(P_{ik} - \frac{\gamma_{ik}}{2} P\right)\sqrt{\gamma}\, \delta g^{ik}\, d\Omega = 0$$

This follows from the fact that $d(\sqrt{\gamma}) = -\frac{1}{2}\sqrt{\gamma}\, g_{ik}\,\delta g^{ik}$ and it is shown in any standard book in General relativity that $\int \delta R_{ik}\, d\Omega = 0$. For the empty space-time solution one has

$$P_{ik} - \frac{g_{ik}}{2} P = 0 \tag{50}$$

In feeding the metric of (34) into this, and rescaling according the previous calculation it is a trivial matter to see that one gets an equation identical to that of (49)

It is in fact rather easy to take the " corner " theory a bit further by assuming a general vacuum metric with a corner matrix as follows

$$\gamma_{ik} = \left(\begin{array}{c|c} g_{ik} & 0 \\ \hline 0 & D_{ab} \end{array}\right) \qquad \gamma^{ik} = \left(\begin{array}{c|c} g^{ik} & 0 \\ \hline 0 & D^{ab} \end{array}\right) \qquad \begin{array}{l} i,k = 1-4 \\ a,b = 5-n \end{array} \tag{51}$$

the connection is almost identical to that of (35), and the interesting bits are

$$\Gamma^a_{b\mu} = \frac{D^{ac}}{2} D_{bc,\mu} \qquad \Gamma^\mu_{ab} = -\frac{g^{\mu\nu}}{2} D_{ab,\nu} \tag{52}$$

* See for example A. Licnerowicz "Théories Relativistes de la Gravitation et de l'Électromagnétisme" 1955, Masson et Cie. The book also contains an elegant treatise on Kaluza theories using methods devised by E. Cartan.

In a completely analogous computation one gets:

$$R^i_{klm} = \hat{R}^i_{klm} \qquad R^i_{kam} = R^a_{klm} = 0$$

$$R^a_{kbm} = -\frac{D^{ac}}{2} D_{bc,m} + \Gamma^n_{km} \frac{D^{ac}}{2} D_{bc,n} - \frac{1}{4} D^{ac}_{,m} D_{bc,k}$$

$$R^i_{alb} = -\frac{1}{2} \left(g^{im} D_{ab,m} \right)_{,l} - \frac{1}{2} \Gamma^i_{nl} g^{nm} D_{ab,m} + \frac{1}{4} g^{im} D^{cd} D_{cb,m} D_{ad,c}$$

$$R_{km} = \hat{R}_{km} - \frac{1}{2} D^{ac} \left(D_{ac,k} \right)_{;m} - \frac{1}{4} D^{ac}_{,m} D_{ac,k}$$

$$R_{am} = 0$$

$$R_{ab} = -\frac{1}{2} \left(g^{lm} D_{ab,m} \right)_{,l} - \frac{1}{2} \Gamma^l_{nl} g^{nm} D_{ab,m} + \frac{1}{4} g^{lm} D_{cb,m} D_{ad,l} D^{cd}$$

In forming the action as

$$R = g^{km} R_{km} + D^{ab} R_{ab}$$

One gets

$$R = \hat{R} - D^{ab} D_{ab,k} - \frac{1}{2} D_{cb,m} D^{cd} D_{ad,l}$$

Now, since $D^{ab} \rightarrow D^{ab}/D$, where $D = \det(D_{ab})$ and

$$[D^{ab}/D]_{,l} = D^{ab}_{;k}/D - D^{ab} D_{,k}/D^2 = D^{ab}_{,k}/D + D^{ab}_{,k} D_{ab}^{,k}/D$$

one shows easily that the action S is

$$S = \int \sqrt{g} \sqrt{D}\, R^{(4)}$$

which is identical to (39)!

One would therfore expect (this is pure guesswork) that this could be rescaled to obtain an action of the type

$$R^{(4)} + F\left(D^{ab} D_{ab,l} \right)$$

and that this solution would satisfy (50).

CONCLUSIONS AND FINAL REMARKS

In this derivation of the Kaluza-Klein theory something obviously has gone wrong. The most important defect is that the Einstein tensor, as formed in the intermediate step leading to the action, contains the vectorpotentials themselves, which is a non-physical object. If there is a slip in the calculations somewhere, it probably lies in the computation of the Cristoffel symbols. It is pleasing, however, that the gravitational and electromagnetic action are produced in a straightforward (but tedious !) manner. It is quite clear, however, that in performing calculations of of this kind in "real life", they are bound to become infinitely more complicated. For example, in rescaling the full 5-d metric - letting everything depend on space-time - the computation would make the original Kaluza-Klein tensor excersise look like something from a 6:th form math. book. There is, however, a smarter and less time/space consuming way of doing all this. In 1928,* a French mathematician, Émile Cartan, published a treatise on Riemannian geometry that reduced the all complicated mass of letters and indeces to handling a simple algebra called exterior algebra. the operations involved are distantly related to the cross product in ordinary geometry, and makes the possibility of making a computational error almost nonexistent.

All the modern Kaluza theories have taken quantum mecanical effects in to account in order to explain the other two forces of nature. This problem, relativity being a classical theory, was realised by Kaluza himself. In his final remark in his paper he concludes " In any case, every ansatz which claims universal validity, is threatened by the sphinx of modern physics, quantum theory."

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* "Leçons sur la Géométrie des Espaces de Riemann"
Gauthier-Villars, Paris, France.

APPENDIX I

Calculations with the Kaluza-Klein metric.

Kaluza – Klein metric: (1)

$$\gamma_{ik} = \left(\begin{array}{c|c} a & abA_k \\ \hline abA_i & g_{ik} + ab^2 A_i A_k \end{array}\right) \qquad \gamma^{ik} = \left(\begin{array}{c|c} \frac{1}{a} + b^2 A^k A_k & -bA^k \\ \hline -bA^i & g^{ik} \end{array}\right)$$

$a = \text{const}$, $\det(\gamma_{ik}) = a \cdot g$

all indices in $1 \ldots 4$

$$\Gamma^0_{00} = \frac{\gamma^{00}}{2}(-\gamma_{00,0}) = 0 \qquad \Gamma^\alpha_{00} = 0 \qquad \Gamma^{\alpha(4)}_{\beta\gamma} = \hat{\Gamma}^\alpha_{\beta\gamma} \ (\text{notation})$$

$$\Gamma^\alpha_{0\beta} = \frac{\gamma^{\alpha\gamma}}{2}(\gamma_{\gamma 0,\beta} - \gamma_{0\beta,\gamma}) = \frac{ab}{2} F^\alpha{}_\beta = \Gamma^\alpha_{\beta 0}$$

$$\Gamma^0_{0\beta} = \frac{\gamma^{0\gamma}}{2}(\gamma_{\gamma 0,\beta} - \gamma_{\beta\gamma,0}) = -\frac{ab^2 A^\gamma}{2} F_{\gamma\beta} = \Gamma^0_{\beta 0}$$

$$\Gamma^0_{\alpha\beta} = \frac{\gamma^{0\gamma}}{2}(\gamma_{\gamma\alpha,\beta} + \gamma_{\beta\gamma,\alpha} - \gamma_{\alpha\beta,\gamma}) + \frac{\gamma^{00}}{2}(\gamma_{0\alpha,\beta} + \gamma_{0\beta,\alpha})$$

$$= -bA_\gamma \Gamma^\gamma_{\alpha\beta} - \frac{ab^3}{2} A^\gamma \big(A_\gamma A_{\alpha,\beta} + A_\alpha A_{\gamma,\beta} + A_\beta A_{\gamma,\alpha} + A_\gamma A_{\beta,\alpha} - A_\alpha A_{\beta,\gamma} - A_\beta A_{\alpha,\gamma} + A^\gamma A_\gamma A_{\alpha,\beta} + A^\gamma A_\gamma A_{\beta,\alpha} \big)$$

$$+ \frac{b}{2}(A_{\alpha,\beta} + A_{\beta,\alpha}) = -bA_\mu \hat{\Gamma}^\mu_{\alpha\beta} - \frac{ab^3}{2} A^\gamma (A_\alpha F_{\gamma\beta} + A_\beta F_{\gamma\alpha}) + \frac{b}{2}(A_{\alpha,\beta} + A_{\beta,\alpha})$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{\gamma^{\alpha\mu}}{2}(\ldots\ldots) + \frac{\gamma^{\alpha 0}}{2}(\gamma_{0\beta,\gamma} + \gamma_{\gamma 0,\beta})$$

$$= \hat{\Gamma}^\alpha_{\beta\gamma} + \frac{ab^2}{2}(A_\beta F^\alpha{}_\gamma + A_\gamma F^\alpha{}_\beta) \qquad (\text{analogous calculation as } \Gamma^0_{\alpha\beta})$$

$$R^i{}_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{\alpha l}\Gamma^\alpha_{km} - \Gamma^i_{\alpha m}\Gamma^\alpha_{kl} + \Gamma^i_{0l}\Gamma^0_{km} - \Gamma^i_{0m}\Gamma^0_{kl}$$

(choose a locally Minkowski frame $\Rightarrow \hat{\Gamma}^i_{km} = 0$)

$$= \frac{ab^2}{2}\left[(A_k F^i{}_m)_{,l} + (A_m F^i{}_k)_{,l} - (A_k F^i{}_l)_{,m} - (A_l F^i{}_k)_{,m} \right] +$$

$$+ \frac{a^2 b^4}{4}\left[(A_\alpha F^i{}_l + A_l F^i{}_\alpha)(A_k F^\alpha{}_m + A_m F^\alpha{}_k) - (A_\alpha F^i{}_m + A_m F^i{}_\alpha)(A_k F^\alpha{}_l + A_l F^\alpha{}_k) \right]$$

$$+ \frac{ab}{2} F^i{}_l \left(-\frac{ab^3}{2} A^\mu [A_k F_{\mu m} + A_m F_{\mu k}] + \frac{b}{2}(A_{k,m} + A_{m,k}) \right) -$$

$$- \frac{ab}{2} F^i{}_m \left(-\frac{ab^3}{2} A^\mu (A_k F_{\mu l} + A_l F_{\mu k}) + \frac{b}{2}(A_{k,l} + A_{l,k}) \right)$$

Sorting out the derivatives:

$$\frac{ab^3}{2}\Big[F^i_k F_{me} + A_k(F^i_{m,e} - F^i_{e,m}) + A_{k,e}F^i_m + A_m F^i_{k,e} - A_{k,m}F^i_e -$$

$$- A_e F^i_{k,m} + \frac{1}{2}F^i_e(A_{k,m} + A_{m,k}) - \frac{1}{2}F^i_m(A_{k,e} + A_{e,k})\Big] = \text{(replace , by ;)}$$

$$= \frac{ab^3}{4}\left[F^i_m F_{ke} + F^i_e F_{mk} + 2F^i_k F_{me}\right] + \frac{ab^3}{2}\Big\{A_k(F^i_{m;e} - F^i_{e;m})$$

$$+ A_m F^i_{k;e} - A_e F^i_{k;m}\Big\}$$

The rest of terms not involving derivatives:

$$\frac{a^2b^4}{4}\Big(-F^i_e F_{\mu m}A^\mu A_k - F^i_e A^\mu A_m F_{\mu k} + F^i_m A^\mu A_k F_{\mu e} + F^i_m F_{\mu k}A^\mu A_e$$

$$- A_\mu A_k F^i_m F^\mu_e - A_\mu A_e F^i_m F^\mu_k - A_m A_k F^i_\mu F^\mu_e - A_m A_e F^i_\mu F^\mu_k$$

$$+ A_\mu A_k F^i_e F^\mu_m + A_\mu A_m F^i_e F^\mu_k + A_e A_k F^i_\mu F^\mu_m + A_e A_m F^i_\mu F^\mu_k\Big)$$

$$= \frac{a^2b^4}{4}A_k F^i_\mu(A_e F^\mu_m - A_m F^\mu_e)$$

changing order of indices.

$$\Rightarrow \underline{R^i_{kem}} = \hat{R}^i_{kem} + \frac{ab^3}{4}(F^i_m F_{ke} - F^i_e F_{km}) - \frac{ab^3}{2}F^i_k F_{em} +$$

$$\frac{ab^3}{2}\left[A_k(F^i_{m;e} - F^i_{e;m}) + A_m F^i_{k;e} - A_e F^i_{k;m}\right] - \frac{a^2b^4}{4}A_k F^i_\mu(A_e F^\mu_m - A_m F^\mu_e)$$

$$\underline{R^i_{ke0}} = \Gamma^i_{k0,e} + \Gamma^i_{ae}\Gamma^a_{k0} - \Gamma^i_{a0}\Gamma^a_{ke} + \Gamma^i_{0e}\Gamma^0_{k0}$$

$$= \frac{ab}{2}F^i_{k;e} + \frac{a^2b^3}{4}\left[(A_a F^i_e + A_e F^i_a)F^a_k - (A_k F^a_e + \cancel{A_e F^a_k})F^i_a - \cancel{F^i_e F_{\mu k}A^\mu}\right]$$

$$= \underline{\frac{ab}{2}F^i_{k;e} - \frac{a^2b^3}{4}A_k F_{ae}F^{ia}}$$

Minkowski frame ($\Gamma = 0$)

$$\underline{R^0_{k0m}} = -\Gamma^0_{k0,m} - \Gamma^0_{a0}\Gamma^a_{km} - \Gamma^0_{am}\Gamma^a_{0k} - \Gamma^0_{0m}\Gamma^0_{k0} = \frac{ab^3}{2}(A^\mu F_{\mu k})_{,m} -$$

$$- \frac{a^2b^4}{4}\Big[A^\mu F_{\mu a}(A_k F^a_m - \cancel{A_m F^a_k}) + A^\mu A^a F_{\mu m}F_{ak} \mp F^a_k A^\mu(\cancel{A_a F_{\mu m}} + \cancel{A_m F_{\mu a}})\Big]$$

$$+ \frac{ab^3}{4}F^a_k(A_{a,m} - A_{m,a}) = \underline{\frac{ab^3}{2}A_a F^a_{k;m} + \frac{ab^3}{4}F^a_k F_{am} - \frac{a^2b^4}{4}A_k A_a F_{\mu m}F^{a\mu}}$$

$$\underline{R^i_{0k0}} = -\Gamma^i_{a0}\Gamma^a_{k0} = -\frac{a^2b^2}{4}F^i_a F^a_k = \underline{\frac{a^2b^2}{4}F^{ia}F_{ka}}$$

Contraction (forming the Ricci tensor)

$$\underline{R_{km}} = R^{l}{}_{klm} + R^{0}{}_{k0m} = \underline{\hat{R}_{km} - \frac{1}{2} ab^2 F^{a}{}_{k} F_{am} +}$$

$$\underline{\frac{ab^2}{2}\left(A_k F^{l}{}_{m;l} + A_m F^{l}{}_{k;l}\right) + \frac{a^2b^4}{4} A_k A_m F^{al} F_{al}}$$

$$\underline{R_{k0}} = R^{l}{}_{kl0} = \frac{ab}{2} F^{l}{}_{k;l} - \frac{a^2b^3}{4} A_k F_{al} F^{al} = \underline{\frac{ab}{2} F^{l}{}_{k;l} + \frac{a^2b^3}{4} A_k F^{al} F_{al}}$$

$$\underline{R_{00}} = \frac{a^2b^2}{4} F^{al} F_{al}$$

Finally forming the curvature scalar:

$$R = g^{km} R_{km} + 2\gamma^{0k} R_{0k} + \gamma^{00} R_{00} = \Big(\hat{R} - \frac{1}{2} ab^2 F^{al} F_{al} + ab^2 A^k F^{l}{}_{k;l} +$$

$$+ \frac{a^2b^4}{4} A^k A_k F^{al} F_{al}\Big) - \Big(ab^2 A^k F^{l}{}_{k;l} + \frac{a^2b^4}{2} A^k A_k F^{al} F_{al}\Big) +$$

$$+ \left(\frac{1}{b^2} + b^2 A^k A_k\right) \frac{a^2b^2}{4} F^{al} F_{al} = \underline{\underline{\hat{R} - \frac{ab^2}{4} F^{al} F_{al}}}$$

APPENDIX V

Calculations with the "corner" function.

The curvature components:

$$\gamma_{ik} = \left(\begin{array}{c|c} \varphi(x) & 0 \\ \hline 0 & g_{ik} \end{array}\right) \qquad \gamma^{ik} = \left(\begin{array}{c|c} \frac{1}{\varphi(x)} & 0 \\ \hline 0 & g^{ik} \end{array}\right)$$

$$\Gamma^0_{00} = \Gamma^0_{\mu\nu} = \Gamma^\mu_{\nu 0} = 0 \qquad \Gamma^\mu_{00} = -\frac{1}{2}\varphi^{,\mu} \qquad \Gamma^0_{0\mu} = \frac{1}{2\varphi}\varphi_{,\mu}$$

$$\Gamma^\mu_{\nu\gamma} = \hat{\Gamma}^\mu_{\nu\gamma}$$

$$R^i_{klm} = \hat{R}^i_{klm}$$

$$R^0_{k0m} = -\Gamma^0_{k0,m} - \Gamma^c_{km}\Gamma^0_{c0} + \Gamma^0_{a0}\Gamma^a_{km} = -\frac{1}{2}\left(\frac{1}{\varphi}\varphi_{,k}\right)_{;m} - \frac{1}{4\varphi^2}\varphi_{,k}\varphi_{,m}$$

$$= -\frac{1}{2\varphi}\varphi_{;k;m} + \frac{1}{4\varphi^2}\varphi_{;k}\varphi_{;m}$$

$$R^i_{0i0} = -\frac{1}{2}\varphi^{;i}{}_{;i} + \frac{1}{4\varphi}\varphi_{;i}\varphi^{;i}$$

$$R_{km} = R^i_{kim} + R^0_{k0m} = \hat{R}_{km} - \frac{1}{2\varphi}\varphi_{;k;m} + \frac{1}{4\varphi^2}\varphi_{;k}\varphi_{;m}$$

$$R_{0m} = 0$$

$$R_{00} = R^i_{0i0} = -\frac{1}{2}\varphi^{;k}{}_{;k} + \frac{1}{4\varphi}\varphi^{;k}\varphi_{;k}$$

$$R = g^{km}R_{km} + \varphi^{-1}R_{00} = \hat{R} - \frac{\varphi^{;k}{}_{;k}}{\varphi} + \frac{1}{2\varphi^2}\varphi_{;k}\varphi^{;k}$$

$$S = \int \sqrt{g}\sqrt{\varphi}\left(\hat{R} - \frac{\varphi^{;k}{}_{;k}}{\varphi} + \frac{1}{2\varphi^2}\varphi^{;k}\varphi_{;k}\right) d\Omega$$: Partial integration of the second term yields:

$$-\int \sqrt{g}\sqrt{\varphi}\,\frac{\varphi^{;k}{}_{;k}}{\varphi}\, d\Omega = -\int \frac{(\sqrt{g}\,\varphi^{;k})_{,k}}{\sqrt{\varphi}}\, d\Omega = -\int \sqrt{g}\,\frac{\varphi^{;k}\varphi_{,k}}{\varphi^{3/2}} = -\int \sqrt{g}\sqrt{\varphi}\,\frac{\varphi^{;k}\varphi_{,k}}{\varphi^2}$$

$$\Rightarrow S = \int \sqrt{g}\sqrt{\varphi}\,\hat{R}\, d\Omega.$$

APPENDIX (III)

calculations with the rescaled corner function

+ a simple proof of the invariance of

ds^2 under (23)

Rescaling of "corner" metric: ($\gamma_{ik} \to \varphi^p g_{ik}$) (1) 2)

$$\gamma_{ik} = \left(\begin{array}{c|c} \varphi^{p+1} & 0 \\ \hline 0 & \varphi^p g_{ik}\end{array}\right) \quad \gamma^{ik} = \left(\begin{array}{c|c} \varphi^{-1-p} & 0 \\ \hline 0 & \varphi^{-p} g^{ik}\end{array}\right)$$

(all indices from 1–4)

$$\Gamma^0_{00} = \Gamma^0_{\mu\nu} = \Gamma^\mu_{\nu 0} = 0 \qquad \Gamma^\mu_{00} = -\frac{(p+1)}{2}\varphi^{,\mu} \qquad \Gamma^0_{0\mu} = \frac{(p+1)}{2\varphi}\varphi_{,\mu}$$

$$\Gamma^\mu_{\nu\gamma} = \hat\Gamma^\mu_{\nu\gamma} + \frac{p}{2\varphi}\left(\delta^\mu_\nu \varphi_{,\gamma} + \delta^\mu_\gamma \varphi_{,\nu} - g_{\nu\gamma}\varphi^{,\mu}\right).$$ Calculation in Minkowski frame:

$$R^i_{kem} = \hat R^i_{kem} + \frac{p}{2}\left(\delta^i_m \left(\frac{\varphi_{,k}}{\varphi}\right)_{,e} - \delta^i_e\left(\frac{\varphi_{,k}}{\varphi}\right)_{,m} + g_{ke}\left(\frac{\varphi^{,i}}{\varphi}\right)_{,m} - g_{km}\left(\frac{\varphi^{,i}}{\varphi}\right)_{,e}\right) + \frac{p^2}{4\varphi^2}\Big(\delta^i_m \varphi_{,e}\varphi_{,k} - \delta^i_e\varphi_{,m}\varphi_{,k} + g_{ke}\varphi_{,m}\varphi^{,i} - g_{km}\varphi_{,e}\varphi^{,i}$$

$$+ \delta^i_e\varphi_{,m}\varphi_{,k} - \delta^i_m \varphi_{,e}\varphi_{,k} + \delta^i_e\varphi_{,m}\varphi_{,k} - \delta^i_m\varphi_{,e}\varphi_{,k} - \delta^i_m g_{ke}\varphi^{,n}\varphi_{,n} -$$

$$- \delta^i_e g_{km}\varphi^{,n}\varphi_{,n} - \varphi^{,i}\varphi_{,m}g_{ke} - \varphi^{,i}\varphi_{,k}g_{em} + g_{km}\varphi^{,i}\varphi_{,e} - g_{km}\varphi^{,i}\varphi_{,e} +$$

$$+ g_{mk}\varphi^{,i}\varphi_{,e} - g_{ke}\varphi^{,i}\varphi_{,m}\Big]$$

$$= \frac{p}{2}\left(\delta^i_m\left(\frac{\varphi_{,k}}{\varphi}\right)_{;e} - \delta^i_e\left(\frac{\varphi_{,k}}{\varphi}\right)_{,m} + g_{ke}\left(\frac{\varphi^{,i}}{\varphi}\right)_{,m} - g_{km}\left(\frac{\varphi^{,i}}{\varphi}\right)_{,e}\right)$$

$$+\frac{p^2}{4\varphi^2}\Big[\delta^i_e\varphi_{,k}\varphi_{,m} - \delta^i_m\varphi_{,k}\varphi_{,e} + g_{km}\varphi^{,i}\varphi_{,e} - g_{ke}\varphi^{,i}\varphi_{,m} +$$

$$+ \delta^i_m g_{ke}\varphi^{,n}\varphi_{,n} - \delta^i_e g_{km}\varphi^{,n}\varphi_{,n}\Big]$$

$$R^0_{k0m} = -\Gamma^0_{k0,m} + \Gamma^n_{km}\Gamma^0_{n0} - \Gamma^n_{0m}\Gamma^0_{nk} = -\frac{(p+1)}{2\varphi}\varphi_{,km} + \frac{(p+1)^2}{4\varphi^2}\varphi_{,k}\varphi_{,m}$$

$$- \frac{p(p+1)}{4\varphi^2} g_{km}\varphi^{,n}\varphi_{,n}$$

The Ricci tensor: $$R_{km} = \hat R_{km} - \frac{p}{2}\left[2\left(\frac{\varphi_{,k}}{\varphi}\right)_{,m} + g_{km}\left(\frac{\varphi^{,n}}{\varphi}\right)_{,n}\right] +$$

$$+\frac{p^2}{4\varphi^2}\left[2\varphi_{,k}\varphi_{,m} - 2g_{km}\varphi^{,n}\varphi_{,n}\right] - \frac{(p+1)}{2\varphi}\varphi_{,km} + \frac{(p+1)^2}{4\varphi^2}\varphi_{,k}\varphi_{,m} - \frac{p(p+1)}{4\varphi^2}g_{km}\varphi^{,n}\varphi_{,n}$$

$$= \frac{(3p^2+6p+1)}{4}\frac{\varphi_{,k}\varphi_{,m}}{\varphi^2} + \frac{(p-3p^2)}{4}\frac{g_{km}}{\varphi^2}\varphi^{,n}\varphi_{,n} - \frac{\varphi_{,km}}{2\varphi}(3p+1) - \frac{p}{2}g_{km}\varphi^{,n}{}_{;n}$$

putting $p=-\frac{1}{3}$ (since $S=\int\sqrt{g}\,\varphi^{\frac{3p}{2}+\frac{1}{2}-p}(\ldots\ldots)\,d\Omega =$ ⑦

$= \int\sqrt{g}\,(\ldots\ldots)\,d\Omega$ if $p=-\frac{1}{3}$)

this turns out to be

$$R_{km} = -\frac{1}{6}\frac{\varphi_{,k}\varphi_{,m}}{\varphi^2} - \frac{1}{6}g_{km}\frac{\varphi^{,a}\varphi_{,a}}{\varphi^2} + \frac{1}{6}g_{km}\varphi^{,a}{}_{;a}$$

partial integration (done in the text) of the 3rd term

$$R_{km} = -\frac{1}{6}\frac{\varphi_{,k}\varphi_{,m}}{\varphi^2}$$

$$R^{i}{}_{0i0} = \Gamma^{i}_{00,i} + \Gamma^{a}_{00}\Gamma^{i}_{ai} - \Gamma^{i}_{a0}\Gamma^{a}_{0i} = -\frac{(p+1)}{2}\varphi^{,i}{}_{;i} - \frac{p(p+1)}{4\varphi}\delta^{i}_{i}\varphi_{,a}\varphi^{,a} + \frac{(p+1)^2}{4\varphi}\varphi^{,i}\varphi_{,i}$$

$$R_{00}(p=-\tfrac{1}{3}) = R^{i}{}_{0i0}(p=-\tfrac{1}{3}) = -\frac{1}{3}\varphi^{,a}{}_{;a} + \frac{1}{3}\varphi^{,a}\varphi_{,a}$$

Forming the curvature scalar: partial integration ↓

$$R = g^{km}R_{km} + \varphi^{-1}R_{00} = -\frac{1}{2}\frac{\varphi^{,a}\varphi_{,a}}{\varphi^2} + \frac{1}{3}\frac{\varphi^{,a}{}_{;a}}{\varphi} + \hat{R} = \underline{\underline{\hat{R} - \frac{1}{6}\frac{\varphi^{,a}\varphi_{,a}}{\varphi^2}}}$$

Some details of the integrations involved in the text

Proof of invariance of $ds^2 = \left(\gamma_{ik} - \frac{\gamma_{0i}\gamma_{0k}}{\gamma_{00}}\right)dx^i dx^k$

under transf^{m} ㉓: $\quad x^0 = x^{\bar 0} + f_0(x^{\bar i})$

$\quad x^i = f_i(x^{\bar i}) \qquad i = 1\cdots 4$

Notation: $\frac{\partial x^i}{\partial x'^{\bar i}} = \Lambda^{i}{}_{\bar i}$ $\qquad \gamma_{00} = \text{const}$

$$\gamma_{\bar i\bar k} = \Lambda^{i}{}_{\bar i}\Lambda^{k}{}_{\bar k}\gamma_{ik}$$

$$\gamma_{0\bar i}\gamma_{0\bar k} = \underset{\delta^s_0}{\Lambda^{s}{}_{0}}\Lambda^{i}{}_{\bar i}\underset{\delta^p_0}{\Lambda^{p}{}_{0}}\Lambda^{k}{}_{\bar k}\gamma_{si}\gamma_{pk} = \Lambda^{i}{}_{\bar i}\Lambda^{k}{}_{\bar k}\gamma_{0i}\gamma_{0k}$$

$$dx^{\bar i}dx^{\bar k} = \Lambda^{\bar i}{}_{r}\Lambda^{\bar k}{}_{s}\,dx^r dx^s$$

$\delta^i_r \quad \delta^k_s$

$$\left(\gamma_{\bar i\bar k} - \frac{\gamma_{0\bar i}\gamma_{0\bar k}}{\gamma_{00}}\right)dx^{\bar i}dx^{\bar k} = \Lambda^{i}{}_{\bar i}\Lambda^{k}{}_{\bar k}\Lambda^{\bar i}{}_{r}\Lambda^{\bar k}{}_{s}\left(\gamma_{ik} - \frac{\gamma_{0i}\gamma_{0k}}{\gamma_{00}}\right)dx^r dx^s$$

$$= \left(\gamma_{ik} - \frac{\gamma_{0i}\gamma_{0k}}{\gamma_{00}}\right)dx^i dx^k$$

$(k \geq 1)$